On the Additive Structure of the Inverses of Banded Matrices

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ABSTRACT

The additive structure of the inverses of banded matrices is investigated. Under certain conditions, the inverse of a $(2k+1)$-diagonal symmetric banded matrix can be expressed as a sum of k symmetric matrices belonging to the class of inverses of symmetric irreducible tridiagonal matrices. In the nonsymmetric case, a more complicated structure is obtained. Applications are mentioned for the resolution of constant coefficient banded linear systems in VLSI models.

1. INTRODUCTION

Many authors have investigated the problem of characterizing the inverses of banded matrices. The structure of the inverses of tridiagonal matrices is now well known [2, 4–6, 9, 10]. In a recent paper, Barrett and Feinsilver [3] established necessary and sufficient conditions for a matrix to be the inverse of a banded matrix. These conditions involve the vanishing of certain non-principal minors of the matrix.

In this paper, the results of [3] are used to obtain an additive structure for the inverses of banded matrices. The central idea is that, in many cases, the inverse of a $(2k+1)$-diagonal matrix can be modified to form a matrix of rank k.

The results have a practical application in the VLSI implementation of linear transformations solving banded systems with constant coefficient matrices (e.g. the systems derived by the discretization of differential equations).

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2. PRELIMINARIES

Let us recall some results on the inverses of banded matrices.

Let $\mathscr{M}_n$ denote the class of symmetric $n \times n$ matrices with the following structure:

$$M \in \mathscr{M}_n \quad \Leftrightarrow \quad M = (m_{ij}), \quad m_{ij} = \begin{cases} u_i v_j, & i \leqslant j, \\ u_j v_i, & i > j. \end{cases}$$

Let T be an $n \times n$ symmetric irreducible nonsingular tridiagonal matrix. Then it is known [4, 6, 10] that

$$T^{-1} \in \mathscr{M}_n$$

and

$$\det(T^{-1}) = \frac{1}{\det(T)} = u_1 v_n \prod_{k=1}^{n-1} (u_{k+1} v_k - u_k v_{k+1}).$$

Moreover, any nonsingular matrix in $\mathscr{M}_n$ has a tridiagonal inverse.

Let E_n be the upper triangular $n \times n$ matrix defined as follows:

$$E_n = (e_{ij}), \qquad e_{ij} = \begin{cases} 1 & \text{if} \quad i \leqslant j, \\ 0 & \text{if} \quad i > j, \end{cases}$$

and let $L_n = E_n^T - I$. The generic element M of $\mathscr{M}_n$ can be written

$$M = (\mathbf{u}\mathbf{v}^T) \circ E_n + (\mathbf{v}\mathbf{u}^T) \circ L_n,$$

where $\circ$ denotes the Hadamard product and $\mathbf{u}, \mathbf{v}$ are n-vectors.

Definition [3]. A matrix R has vanishing super- (sub-) p-minors if the minor formed by the rows $i_1, i_2, \dots, i_p$ and the columns $j_1, j_2, \dots, j_p$ vanishes for all choices of indices with $i_1 < i_2 < \cdots < i_p$, $j_1 < j_2 < \cdots < j_p$, and $j_1 > i_p - p + 1$ $(i_1 > j_p - p + 1)$.

Lemma 2.1. *A matrix B is $(2k+1)$-diagonal if and only if B^{-1} has vanishing super- and sub-$(k+1)$-minors.*

Proof. This lemma is a reformulation of a theorem of [1]; see also [3]. ■

DEFINITION. A nonsingular $(2k+1)$-diagonal matrix is said to be proper if any submatrix obtained by deleting k consecutive rows and k consecutive columns is nonsingular.

3. RESULTS

The ranks of some nonprincipal submatrices of the inverse of a banded matrix are characterized by the following lemmas.

LEMMA 3.1. *Let B be an $n \times n$ nonsingular $(2k+1)$-diagonal matrix. Any submatrix in the upper (lower) triangular part of its inverse B^{-1} has rank at most k.*

Proof. From Lemma 2.1 any $(k+1)\times(k+1)$ minor in the upper (lower) triangular part of B^{-1} vanishes. ■

LEMMA 3.2. *Let B be an $n \times n$ proper $(2k+1)$-diagonal matrix. Then any submatrix formed by k consecutive rows and columns of its inverse B^{-1} will be nonsingular.*

Proof. Let $1 \leqslant i, j \leqslant n$. Let P be the submatrix of B^{-1} formed by the rows $i,\dots,i+k-1$ and by the columns $j,\dots,j+k-1$, and let Q be the submatrix of B formed by the rows $1,\dots,j-1,j+k,\dots,n$ and by the columns $1,\dots,i-1,i+k,\dots n$.

The submatrix Q is obtained from B by deleting k consecutive rows and columns and is nonsingular because B is proper. From a theorem of Jacobi [8, p. 14], we have

$$|\det(B)||\det(P)| = |\det(Q)|;$$

hence P is nonsingular for any choice of i and j. ■

We can now prove the following theorem on the additive structure of the inverses of symmetric banded matrices.

THEOREM 3.1. *Let B be an $n\times n$ proper $(2k+1)$-diagonal symmetric matrix. Its inverse B^{-1} has the following additive structure:*

$$B^{-1}=\sum_{p=1}^{k} M_p,\qquad M_p\in\mathcal{M}_n.$$

Proof. From Lemma 3.1 any submatrix in the upper corner of B^{-1} has rank at most k. Moreover, from Lemma 3.2 all the submatrices of B^{-1} formed by k consecutive rows and columns are nonsingular.

Let us now partition B^{-1} in the form

$$B^{-1}=\begin{pmatrix} P & Q\\ R & S\end{pmatrix},$$

where Q is a square $k\times k$ matrix. Let U and V be $n\times k$ matrices defined as

$$U=\left(\begin{array}{c} I\\ \hline SQ^{-1}\end{array}\right),\qquad V^T=(P\mid Q).$$

The rank-k matrix $C=UV^T$ has the following structure:

$$C=\begin{pmatrix} P & Q\\ Z & S\end{pmatrix},\qquad Z=SQ^{-1}P.$$

We now consider the $(k+1)\times(k+1)$ submatrices in the upper right corner of B^{-1} and C (say X and Y), namely,

$$X=\left(\begin{array}{c|cccc} p_{1,n-k} & & & & \\ \vdots & & Q & & \\ p_{k,n-k} & & & & \\ \hline r_{1,n-k} & s_{11} & s_{12} & \cdots & s_{1k}\end{array}\right)$$

and

$$Y=\left(\begin{array}{c|cccc} p_{1,n-k} & & & & \\ \vdots & & Q & & \\ p_{k,n-k} & & & & \\ \hline z_{1,n-k} & s_{11} & s_{12} & \cdots & s_{1k}\end{array}\right).$$

From $\det(Q) \neq 0$ and $\det(X) = \det(Y) = 0$ it thus follows that $r_{1,n-k} = z_{1,n-k}$. Analogously, we can prove that $r_{ij} = z_{ij}$ for all $i \leqslant j + k$, so that the upper triangular part of B^{-1} is part of the square matrix C of rank k.

Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the columns of U and V. Then there exists an additive decomposition for C:

$$C = \sum_{p=1}^{k} \mathbf{u}_p \mathbf{v}_p^T.$$

By symmetry we have $B^{-1} = C \circ E_n + C^T \circ L_n$, so that

$$B^{-1} = \sum_{p=1}^{k} \left[\left(\mathbf{u}_p \mathbf{v}_p^T \right) \circ E_n + \left(\mathbf{v}_p \mathbf{u}_p^T \right) \circ L_n \right],$$

and the theorem is proved. ∎

More generally, the following properties of the structure of nonsymmetric banded matrices can be established.

THEOREM 3.2. *Let B be an $n \times n$ proper $(2k+1)$-diagonal matrix. Then its inverse B^{-1} has the following additive structure:*

$$B^{-1} = \sum_{p=1}^{k} \left[\left(\mathbf{u}_p \mathbf{v}_p^T \right) \circ E_n + \left(\mathbf{x}_p \mathbf{y}_p^T \right) \circ L_n \right]$$

with $\mathbf{u}_p, \mathbf{v}_p, \mathbf{x}_p, \mathbf{y}_p$ n-vectors, $p = 1, 2, \ldots, k$.

Proof. The proof is similar to that of Theorem 3.1 where symmetry was used only in the last paragraph. ∎

THEOREM 3.3. *Let B be an $n \times n$ nonsingular $(2k+1)$-diagonal matrix. Any leading or lagging nonsingular principal submatrix of B^{-1} is the inverse of a $(2k+1)$-diagonal matrix.*

Proof. The matrices obtained by deleting the first few rows and columns of B^{-1} or the last few rows and columns of B^{-1} still satisfy the conditions of Lemma 2.1. ∎

THEOREM 3.4. *Let B be an $n \times n$ nonsingular $(2k+1)$-diagonal matrix with nonsingular principal submatrices. Then its inverse B^{-1} has the following structure:*

$$B^{-1} = \begin{pmatrix} C^{-1} & X \\ Y & D^{-1} \end{pmatrix},$$

where C and D are square $(2k+1)$-diagonal matrices of dimension $p \times p$ and $(n-p) \times (n-p)$ respectively (with $k < p < n-k$). Moreover, X and Y have rank at most k.

Proof. The proof directly follows from Theorem 3.3 and Lemma 2.1. ■

Note that Theorem 3.4 can be recursively applied to matrices C and D provided that C and D have nonsingular principal submatrices. This is the basis of the application presented in the following section.

It would be interesting to determine when a sum of matrices belonging to $\mathcal{M}_n$ has banded inverse. Sufficient conditions can be stated for $k = 2$.

THEOREM 3.5. *Let M and N be nonsingular matrices in $\mathcal{M}_n$. Let M^{-1} and $-N^{-1}$ have the same off-diagonal elements and different diagonal elements (i.e., $M^{-1} + N^{-1}$ is diagonal nonsingular). Then $M+N$ is nonsingular and $(M+N)^{-1}$ is pentadiagonal.*

Proof. From the hypothesis we can write the expression

$$P = M^{-1} - M^{-1}RM^{-1} \qquad \text{with} \quad R^{-1} = M^{-1} + N^{-1},$$

which is the Woodbury formula [8, p. 124] for the inverse of $M+N$. Then $M+N$ is nonsingular and $P = (M+N)^{-1}$ is pentadiagonal. ■

EXAMPLE 3.1. Let

$$M = \begin{pmatrix} -2.5 & -7 & 6 & 4 & -8 \\ -7 & -21 & 18 & 12 & -24 \\ 6 & 18 & -15 & -10 & 20 \\ 4 & 12 & -10 & -7 & 14 \\ -8 & -24 & 20 & 14 & -27 \end{pmatrix}$$

and

$$N=\begin{pmatrix} -1 & -1 & 0 & 1 & 2 \\ -1 & -0.5 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & -0.5 & -1 \\ 1 & 0.5 & -0.5 & -1.25 & -2.5 \\ 2 & 1 & -1 & -2.5 & -4 \end{pmatrix}.$$

Then we have

$$M^{-1}=\begin{pmatrix} -6 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix} \quad \text{and}$$

$$N^{-1}=\begin{pmatrix} 1 & -2 & 0 & 0 & 0 \\ -2 & 2 & -2 & 0 & 0 \\ 0 & -2 & 3 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 1 \end{pmatrix}.$$

Finally,

$$M^{-1}+N^{-1}=\begin{pmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

and

$$(M+N)^{-1}=\frac{1}{60}\begin{pmatrix} -8 & -64 & -80 & 0 & 0 \\ -64 & 28 & 50 & -60 & 0 \\ -80 & 50 & -83 & 66 & -48 \\ 0 & -60 & 66 & -132 & 36 \\ 0 & 0 & -48 & 36 & -18 \end{pmatrix}.$$

There are singular matrices in the class $\mathcal{M}_n$ obtained by bordering nonsingular matrices of $\mathcal{M}_p$ $(p<n)$ with null entries. Theorem 3.5 can be generalized to take this case into consideration.

EXAMPLE 3.2. Let

$$M=\begin{pmatrix} -2.5 & -7 & 6 & 4 & -8 \\ -7 & -21 & 18 & 12 & -24 \\ 6 & 18 & -15 & -10 & 20 \\ 4 & 12 & -10 & -7 & 14 \\ -8 & -24 & 20 & 14 & -27 \end{pmatrix}$$

and

$$N=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -4 & 0 \\ 0 & -2 & -1 & -2 & 0 \\ 0 & -4 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$U=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q=\begin{pmatrix} -3 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -3 \end{pmatrix}.$$

Then $N=UQU^T$, and

$$M^{-1}=\begin{pmatrix} -6 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}, \qquad Q^{-1}=\begin{pmatrix} 1 & -2 & 0 \\ -2 & 7 & -2 \\ 0 & -2 & 1 \end{pmatrix}.$$

Finally

$$U^TM^{-1}U+Q^{-1}=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and

$$(M+N)^{-1}=\frac{1}{24}\begin{pmatrix} -192 & 24 & -48 & 0 & 0 \\ -24 & 0 & 18 & -12 & 0 \\ -48 & 18 & -83 & 18 & -48 \\ 0 & -12 & 18 & 0 & 24 \\ 0 & 0 & -48 & 24 & -24 \end{pmatrix}.$$

It is worth noting that a result similar to Theorem 3.1 was presented by Bevilacqua and Capovani in [4]. They prove the following theorem.

THEOREM 3.6 [4]. *Let $\mathcal{Q}_{n,k}$ be the class of $n \times n$ matrices Q for which*

$$q_{ij} = \sum_{p=1}^{k} u_{ip} v_{pj}, \qquad \begin{cases} 1 \leqslant i \leqslant k, & 1 \leqslant j \leqslant n; \\ k+1 \leqslant i \leqslant n, & i-k+1 \leqslant j \leqslant n; \end{cases}$$

$$q_{ij} = q_{ji}, \qquad k+1 \leqslant i \leqslant n, \qquad 1 \leqslant j \leqslant i-k,$$

with the additional condition

$$\sum_{p=1}^{k} u_{ip} v_{pj} = \sum_{p=1}^{k} u_{jp} v_{pi}, \qquad |i-j| \leqslant k.$$

Let $\mathcal{Q}^*_{n,k}$ be the class of nonsingular matrices of $\mathcal{Q}_{n,k}$. Then $\mathcal{Q}^*_{n,k}$ is the class of the inverses of the $(2k+1)$-diagonal symmetrical matrices whose elements on the extreme bands are nonzero.

Note that in this case the terms of the additive decomposition can be nonsymmetric, and in general they do not belong to the class $\mathcal{M}_n$.

4. APPLICATIONS

One result given above can be applied to the VLSI implementation of linear transforms whose matrix is the inverse of a constant banded matrix. Such transformations solve the associated banded linear systems which often arise from the discretization of boundary value problems by difference methods.

Let B be an $n \times n$ $(2k+1)$-diagonal positive definite matrix of constants. The system $B\mathbf{x} = \mathbf{c}$ can be solved by an ad hoc VLSI circuit in time:

$$T = 2\lceil \log n \rceil + 2\lceil \log k \rceil + 2 = O(\log n),$$

and area:

$$A = O(nk \log^3 n).$$

Our algorithm consists in the recursive splitting of the matrix B^{-1} as in Theorem 3.4 (with $p = \lceil n/2 \rceil$) and in the corresponding implementation of

VLSI layouts which can be recursively split into two parts with a reciprocal low flow of information.

An interesting special case of linear transformation of this kind derives from the discretization by difference methods of some partial differential equations in rectangular regions. Straightforward application of the algorithm yields the bound $O(n^3 \log^5 n)$ to the area$\times$time^2 product for any differential equation leading to a system with $O(n^2)$ unknowns and a banded coefficient matrix with $O(n)$ diagonals. A detailed discussion of these VLSI designs can be found in [7].

The author wishes to thank an anonymous referee for his essential help in the improvement of the paper.

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Received 24 September 1984; revised 12 August 1985